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Dynamics of a coating film flow on horizontal cylinders with van der Waals forces

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ABSTRACT

In this work we use the lubrication type method, based on the possibility of the separation of longitudinal and transversal length scales, to simplify the analysis of coating film dynamics. We study the dynamics of coating thin films on horizontal stationary and rotating cylinders in general, and when the effect of van der Waals forces is significant. A computer code based on the method of orthogonal collocation is developed and is used for the study of the instabilities of coating thin film flow on cylinders to identify conditions of stable operation. For stationary cylinders, the equations are symmetric and the two spline collocation second-order formulation method is most appropriate. For rotating cylinders, a spline collocation first-order formulation method is most efficient.

As expected in all cases positive van der Waals forces cause rupture of the film with the rupture time of the film decreasing with the increase of van der Waals forces. Gravity, on the other hand, has a stabilizing effect and increases rupture time.

1. Introduction

Instabilities of thin liquid films between a solid substrate and a gas atmosphere have attracted much scientific interest. In most applications of thin films on coating, homogeneity and durability are desired. Applications range from photographic films, paints, adhesives and optical coatings. A recent review can be found in Ref. [1]. To analyze such instabilities a long-wave or lubrication approximation is often used as a very powerful tool especially for low Reynolds number film flows. The possibility of the separation of longitudinal and transversal length scales in the theoretical analysis of the film dynamics has simplified the treatment of thin film systems and led to a lubrication type approximation. The main assumption is thus that the scale for the height of the fluid is much less than its lateral length scales. The lubrication-theory or long-wave-theory approach is based on the asymptotic reduction of the governing equations and boundary conditions to a simplified system which often consists of a single nonlinear partial differential equation giving the dynamics of the local thickness of the film. Several instability mechanisms exist that, by means of different driving forces, may destabilize an initially uniform film.

Burelbach et al. [2] studied the stability of evaporating films on a horizontal plane. They considered the effect of vapor recoil, surface tension, thermocapillarity and van der Waals forces.

Joo et al. [3] studied the same problem but on an inclined plane where gravitational forces are important. They neglected the effect of van der Waals forces.

In coating applications, a thin layer of a liquid film could be resting or flowing on a cylindrical surface. Such a case needs special treatment.

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A cylinder undergoing uniform rotation about a horizontal axis is able to hold a thin coating of liquid, due to the combined effects of liquid viscosity and cylinder rotation. In general, however, the coating on a long right circular cylinder may be subject to instability due to the action of surface tension at the liquid–air interface, and centripetal acceleration.

Analysis of this problem requires an understanding of the interplay between gravitational, rotational, and surface tension effects on the coating.

In determining whether a coating will be retained on a rotating object, it is necessary to account for the tendency of an elongated volume of liquid to break into droplets under the influence of surface tension.

An evolution equation describing the draining of a thin film on a horizontal cylinder was developed by Reisfeld and Bankoff [4]. They took into account gravity, surface tension, thermocapillarity, and long-range molecular forces.

Weidner et al. [5], using simulations of an evolution equation that included higher-order corrections in film thickness, described the dynamic transition from a uniform film coating the exterior of a cylinder to a pendent rivulet, which became wavy before breaking up to form isolated pendent drops. These authors also described how a sufficiently thick film can form an equilibrium nonaxisymmetric collar (a drop that extends around to the top of the tube). At large Bond numbers break-up of a pendent rivulet into drops can occur.

Roy et al. [6] derived a general evolution equation for any curved substrate that included cylinders as a special case.

Evans et al. [7] found that at slow rotation rates, surface tension has a minimal effect over much of the cylinder, yet it is essential in retaining liquid in the vicinity of the drop which forms near the bottom of the cylinder. As the rotation rate is increased, an increasing amount of liquid is distributed around the cylinder, and the drop hanging beneath the cylinder diminishes in size, while rising ever higher on the upward-moving side of the cylinder. At higher speeds, transportation of liquid around the cylinder means that there is relatively little variation in coating thickness around the cylinder.

"Thumping", in which a bulge of liquid is carried around the cylinder many times, can also occur. Because rotation continually carries liquid over the entire coating, there is a nonzero minimum coating thickness even in the asymptotic sense for large times. This is quite unlike the stationary case, where eventually all liquid will drain from the upper parts of the cylinder.

Evans et al. [7] used a perturbation method based on the aspect ratio (reference thin film height over the cylinder radius) being very small for the derivation of the evolution equations. During the transients, at some points along the cylinder circumference the film gets thick and the film surface curvature is large. In this case the profiles may not be accurate enough. Improvements can be made by keeping higher order terms during the perturbation analysis.

We will extend the results presented in Ref. [7] for the thin film flow on the surface of a cylinder for different types of forces to include van der Waals forces.

The numerical solution of the evolution equation is not easy since it includes steep and oscillating profiles. In Refs. [2,3] the finite difference method was used whereas in Ref. [4], a Fourier spectral method was used. In this paper we will use the method of orthogonal collocation [8–10].

It is intended to study different possible behaviors using a long wave nonlinear analysis of growth of the instability in thin films using numerical simulation. The development of an efficient computer code to examine numerically the film thickness profile on a stationary and rotating horizontal cylinder will be one of the objectives of this paper.

2. Problem formulation

We consider a two-dimensional Newtonian thin liquid film of a constant density ρ and viscosity μ, driven by gravity, surface tension and van der Waals forces down a cylinder. Flow variations in the axial direction are not considered.

A liquid layer on such a cylinder drains to the underside and drips off under the influence of gravity but may be maintained by the effects of rotation, and surface tension. The model developed includes the effect of gravity, cylinder rotation surface tension, and van der Waals forces.

A family of steady solutions exists. These range from a pendant droplet hanging near the cylinder underside at very low speeds to solutions which are nearly symmetric in a horizontal plane through the cylinder center at high speeds.

If the cylinder is not rotating, the coating will drain possibly with some defects such as drip marks. Rotation will produce more even coating.

van der Waals intermolecular forces can produce instabilities leading to film ruptures. We consider this problem of thin film rupture driven by van der Waals forces and look for steady state solutions. Small perturbations from these solutions will lead to finite-time rupture. We look for the solutions close to rupture.

The problem geometry is shown in Fig. 1.

The derivation of the evolution equation follows along the same general lines as for a horizontal plane wall, with the added complication of the gravity force's being azimuthally position dependent. We combine the stationary cylinder model of Reisfeld and Bankoff [4] with the rotating cylinder model of Evans et al. [7] and use the dimensionless groups as defined by Reisfeld and Bankoff [4]. The evolution equation thus obtained is

$$h_t + \{h^3[\sin\theta + Bo^{-1}(h_\theta + h_{\theta\theta\theta})] + Ah^{-1}h_\theta + 3MWh + \varepsilon h^3 h_\theta (W^2 - \cos\theta)\}_\theta = 0. \quad (1)$$

The first term is the height dynamics, the second term is the gravity angular force, the third term is due to surface tension, the fourth term is for van der Waals force, the fifth term is for the angular viscous dragging of the liquid by rotation and

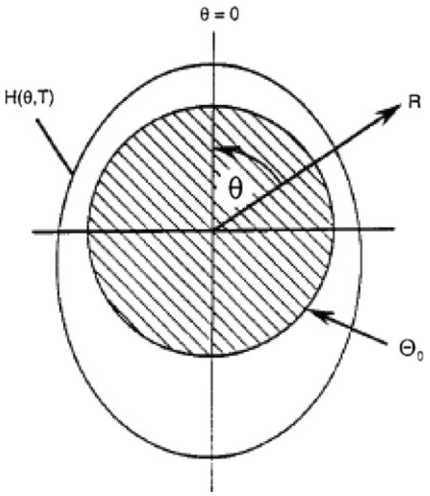

Fig. 1. Schematic representation of the problem of a film on a horizontal cylinder.

Table 1
Problem notation.

ε	Aspect ratio = initial mean film thickness/radius of cylinder = H/R
Bo	Bond number = ratio of gravity forces/mean surface tension = $\rho g R^2/\varepsilon\sigma$
A	Dimensionless Hamaker constant = $A'/2\pi g H^4$
M	Dimensionless viscosity = $\mu/\rho(gR^3)^{0.5}$
W	Dimensionless rotation rate = $\Omega(R/g)^{0.5}$
h	Dimensionless distance; its initial value is unity
Θ	Azimuthal angle measured downward from the vertical line counterclockwise
t	Dimensionless time
μ	Viscosity
ρ	Density
g	Gravitational acceleration
σ	Surface tension
Ω	Cylinder rotation rate
A'	Hamaker constant

the last term includes the centrifugal force and the radial component of the acceleration. The last term is not included in Reisfeld and Bankoff [4] since it can be neglected for very small ε.

We have initially a fixed mass of fluid spread uniformly on a horizontal cylinder, i.e.,

$$h(\theta, 0) = 1. \tag{2}$$

For stationary cylinders, symmetry conditions are applied as

$$h_\theta(0, t) = 0, \quad h_\theta(\pi, t) = 0 \qquad h_{\theta\theta\theta}(0, t) = 0, \quad h_{\theta\theta\theta}(\pi, t) = 0. \tag{3}$$

And generally the periodicity conditions are applied;

$$\frac{\partial h^i(0, t)}{\partial \theta^i} = \frac{\partial h^i(2\pi, t)}{\partial \theta^i} \quad i = 0, 1, 2, 3. \tag{4}$$

The parameters and variables notation are defined in Table 1.

3. Numerical method formulation

Dynamic models for thin liquid falling films require suitable numerical procedures to solve the partial differential equation set that describes the film thickness dynamics. The orthogonal collocation method has been applied for different problems in several chemical engineering applications [8–10]. In this method, a trial function is taken as a series of orthogonal polynomials whose roots are used as collocation points (thus avoiding an arbitrary choice by the user) and the dependent variables become the solution values at these collocation points. The accuracy of the method increases rapidly with the order of the trial function but a first-order approximation usually gives good results. In addition, the method has an additional advantage of reducing by 50% the number of unknown variables if the solution of the model is symmetric. In particular, it is shown that for an nth-order differential equation in one space dimension with two-point derivative boundary conditions, an ideal choice of interior collocation points is the set of zeros of a Jacobi polynomial.

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are defined such that they satisfy the orthogonality conditions [9,10]

$$\int_0^1 w(x)P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx = 0 \quad (n \neq m) \tag{5}$$

and

$$\int_0^1 w(x)P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx = C_n \succ 0 \quad (n = m) \tag{6}$$

where $w(x)$ is the weighting function for the orthogonality conditions. For Jacobi polynomials,

$$w(x) = (1-x)^\alpha x^\beta, \quad \alpha, \beta > -1. \tag{7}$$

Thus $P_n^{(\alpha,\beta)}(x)$ is the orthogonal polynomial of degree n and α, β are the indices of the weighting function. C_n is a constant.

In Ref. [11], many variants of the orthogonal collocation methods have been formulated and tested for thin film flowing problems. It is found that the spline collocation method based on a first-order formulation has shown superiority over the other formulations, however if symmetry exists the two spline collocation method based on a second-order formulation predicts an accurate solution with a minimum number of collocation points.

Here is the description of the two methods;

1. *Multiple spline collocation-first-order formulation method*

In the following we present the formulation of the problem in terms of the orthogonal collocation method.

Let

$$\theta = 2\pi\eta. \tag{8}$$

Eq. (1) can be written in terms of this new variable as follows:

$$\begin{aligned} &h_t + \{h^3[\sin 2\pi\eta + Bo^{-1}(h_\eta/2\pi + h_{\eta\eta\eta}/8\pi^3)] \\ &\quad + Ah^{-1}h_\eta/2\pi - 3MWh + \varepsilon h^3 h_\eta/2\pi(W^2 - \cos 2\pi\eta)\}_\eta/2\pi = 0 \end{aligned} \tag{9}$$

therefore

$$\begin{aligned} &h_t + \{3h^2h_\eta[\sin 2\pi\eta + Bo^{-1}(h_\eta/2\pi + h_{\eta\eta\eta}/8\pi^3)] + h^3[2\pi s\cos 2\pi\eta + Bo^{-1}(h_{\eta\eta}/2\pi + h_{\eta\eta\eta\eta}/8\pi^3)] \\ &\quad - Ah^{-2}h_\eta^2/2\pi - 3MWh_\eta + 3\varepsilon h^2h_\eta^2/2\pi(W^2 + 2\pi\sin 2\pi\eta) \\ &\quad + Ah^{-1}h_{\eta\eta}/2\pi + \varepsilon h^3 h_{\eta\eta}/2\pi(W^2 - \cos 2\pi\eta)\}/2\pi = 0. \end{aligned} \tag{10}$$

Eq. (10) can also be written as a system of first-order differential equations such that

$$\frac{\partial h}{\partial \eta} = u_1 \quad \text{with } h(0) = h_1 \tag{11a}$$

$$\frac{\partial u_1}{\partial \eta} = u_2 \quad \text{with } u_1(1) = u_{1,N+1} \tag{11b}$$

$$\frac{\partial u_2}{\partial \eta} = u_3 \quad \text{with } u_2(0) = u_{2,1} \tag{11c}$$

$$\frac{\partial u_3}{\partial \eta} = u_4 \quad \text{with } u_3(1) = u_{3,N+1}. \tag{11d}$$

The orthogonal collocation method is applied by evaluating the differential equations at the collocation points and setting this residual to zero. The spatial discretization of (10) and (11) by the orthogonal collocation method at the collocation points results in a system of ordinary differential equations over time.

Eq. (11a) in terms of collocation matrices takes the form

$$\sum_{k=1}^{N+1} A'_{j+1,k}h_k = u_{1,j}, \quad j = 1, 2, \ldots, N \tag{12}$$

where $\underset{\sim}{A'}$ is the weight matrix of the first derivative for a boundary point at $x = 0$.

Subroutines for the calculations of derivative weight matrices for orthogonal collocation up to second order are available in Ref. [9]. These are extended to any order in Ref. [10].

Eq. (11b) can be written as follows

$$\sum_{j=1}^{N} A_{i-1,j}u_{1,j} + A_{i-1N+1}u_{1,N+1} = u_{2,i}, \quad i = 2, 3, \ldots, N+1 \tag{13}$$

where $\underset{\sim}{A}$ is the matrix of first derivatives for the boundary point at $x = 1$.

Substituting (12) into (13) leads to

$$\sum_{j=1}^{N}\sum_{k=1}^{N+1} A_{i-1,j}A'_{j+1,k}h_k + A_{i-1N+1}u_{1,N+1} = u_{2,i}, \quad i = 2, 3, \ldots, N+1. \tag{14}$$

But

$$A_{N+2-i,N+2-j} = -A'_{i,k}$$

$$\sum_{k=1}^{N+1} B_{i,k}h_k + A_{i-1N+1}u_{1,N+1} = u_{2,i}, \quad i = 2, 3, \ldots, N \tag{15}$$

where

$$B_{i,k} = -\sum_{j=1}^{N} A_{i-1,j}A_{N+1-j,N+2-k} \quad \begin{array}{l} i = 2, \ldots, N+1 \\ k = 1, \ldots, N+1. \end{array}$$

The third-order derivative (11c) can be written as

$$\begin{aligned} &A'_{i+1,1}u_{2,1} + \sum_{k=2}^{N+1} A'_{i+1,k}u_{2,k} = u_{3,i} \quad i = 1, \ldots, N \\ &-A_{N+1-i,N+1}u_{2,1} - \sum_{k=2}^{N+1}\sum_{j=1}^{N+1} A_{N+1-i,N+2-k}B_{k,j}h_j - \sum_{k=2}^{N+1} A_{N+1-i,N+2-k}A_{k-1,N+1}u_{1,N+1} = u_{3,i}. \end{aligned} \tag{16}$$

The above equation can be expressed as

$$-A_{N+1-i,N+1}u_{2,1} + \sum_{j=1}^{N+1} BB_{i,j}h_j + B1_{i,N+1}u_{1,N+1} = u_{3,i} \tag{17}$$

where

$$BB_{i,j} = -\sum_{k=2}^{N+1} A_{N+1-i,N+2-k}B_{k,j}$$

$$B1_{i,N+1} = -\sum_{k=2}^{N+1} A_{N+1-i,N+2-k}A_{k-1,N+1} \quad \begin{array}{l} i = 1, \ldots, N \\ j = 1, \ldots, N+1. \end{array}$$

The fourth-order derivative (Eq. (11d)) is expressed as

$$\begin{aligned} &A_{i-1,N+1}u_{3,N+1} + \sum_{j=1}^{N} A_{i-1,j}u_{3,j} = u_{4,i} \quad i = 2, \ldots, N+1 \\ &-\sum_{j=1}^{N} A_{i-1,j}A_{N+1-j,N+1}u_{2,1} + \sum_{j=1}^{N}\sum_{k=1}^{N+1} A_{i-1,j}BB_{j,k}h_k + \sum_{j=1}^{N} A_{i-1,j}B1_{j,N+1}u_{1,N+1} = u_{4,i} \end{aligned} \tag{18}$$

or

$$B2_{i,1}u_{2,1} + \sum_{k=1}^{N+1} BBB_{i,k}h_k + BB1_{i.N+1}u_{1,N+1} = u_{4,i} \tag{19}$$

where

$$B2_{i,1} = -\sum_{j=1}^{N} A_{i-1,j}A_{N+1-j,N+1}$$

$$BBB_{i,k} = \sum_{j=1}^{N} A_{i-1,j}BB_{j,k} \quad \begin{array}{l} i = 2, \ldots, N+1 \\ k = 1, \ldots, N+1 \end{array}$$

$$BB1_{i,N+1} = \sum_{j=1}^{N} A_{i-1,j}B1_{j,N+1}.$$

The film thickness equation becomes

$$h_t + \{3h^2u_1[\sin 2\pi\eta + Bo^{-1}(u_1/2\pi + u_3/8\pi^3)] + h^3[2\pi s\cos 2\pi\eta + Bo^{-1}(u_2/2\pi + u_4/8\pi^3)] \\ - Ah^{-2}u_1^2/2\pi - 3MWu_1 + 3\varepsilon h^2u_1^2/2\pi(W^2 + 2\pi\sin 2\pi\eta) \\ + Ah^{-1}u_2/2\pi + \varepsilon h^3u_2/2\pi(W^2 - \cos 2\pi\eta)\}/2\pi = 0. \tag{20}$$

For the new formulation the boundary conditions are defined as follows

$$h(1) - h(0) = \int_0^1 \frac{du_1}{d\eta}d\eta = 0 \\ = \sum_{j=1}^{N} w_j \frac{du_{1,j}}{d\eta} \tag{21a}$$

$$u_1(1) - u_1(0) = \int_0^1 \frac{du_2}{d\eta}d\eta = 0 \\ = \sum_{j=1}^{N} w_j \frac{du_{2,j+1}}{d\eta} = 0 \tag{21b}$$

$$u_2(1) - u_2(0) = \int_0^1 \frac{du_3}{d\eta}d\eta = 0 \\ = \sum_{j=1}^{N} w_j \frac{du_{3,j}}{d\eta} = 0 \tag{21c}$$

$$u_3(1) - u_3(0) = \int_0^1 \frac{du_4}{d\eta}d\eta = 0 \\ = \sum_{j=1}^{N} w_j \frac{du_{4,j+1}}{d\eta} = 0 \tag{21d}$$

where w_j are the weights of the quadrature. We have $N+4$ equations in $N+4$ unknowns; N equations from the satisfaction of the differential equations at N collocation points, and four boundary conditions equations. We have N unknowns in h at N collocation points and 4 boundary points unknown, $h(0), u_1(1), u_2(0), u_3(1)$.

Due to the steepness of the profile close to the rupture points in case of the presence of van der Waals forces, we may resort to the use of the spline collocation method. In this case we divide the $\eta \in [0, 1]$ into m splines. We will redefine η to $\bar{\eta}$ such that $\bar{\eta} = \frac{\eta}{m}$. In the governing equations we will replace $k/2\pi$ by $k/(2\pi m)$ and we will add in each spline the following equations

$$h(k) - h(k-1) - \sum_{j=1}^{N} w_j \frac{du_{i,j+k(N-1)}}{d\eta'} = 0; \quad k = 1, 2, \dots, m \tag{22a}$$

notice that $h(m) = h(0)$

$$u_1(k) - u_1(k-1) - \sum_{j=1}^{N} w_j \frac{du_{2,j+k(N-1)}}{d\eta'} = 0; \quad k = 1, 2, \dots, m \tag{22b}$$

and $u_1(m) = u_1(0)$

$$u_2(k) - u_2(k-1) - \sum_{j=1}^{N} w_j \frac{du_{3,j+k(N-1)}}{d\eta'} = 0; \quad k = 1, 2, \dots, m \tag{22c}$$

and $u_2(m) = u_2(0)$

$$u_3(k) - u_3(k-1) - \sum_{j=1}^{N} w_j \frac{du_{4,j+k(N-1)}}{d\eta'} = 0; \quad k = 1, 2, \dots, m \tag{22d}$$

and $u_3(m) = u_3(0)$.

The number of equations to solve are $n \times m$ governing equations plus $(m + 1) \times 4$ boundary conditions. The number of unknowns is $(n + 4) \times m + 4$ which is equal to the number of equations and the boundary conditions. As we have shown above the collocation method converts the evolution partial differential equation into a set of ordinary differential and algebraic equation. For the resulting system, a DAE (differential–algebraic solver) is needed. We have chosen the DASSL FORTRAN code to solve the system which is coded in FORTRAN. DASSL is freely available in the public domain and it uses backward differentiation formula methods to solve a system of DAEs.

2. *Two spline collocation-second-order formulation method*

For the case of a stationary cylinder where the governing equation (10) is symmetric, another solution scheme is derived to solve Eq. (10) based on recasting the model into two second-order differential equations. The second derivative of the thickness is defined as

$$\frac{\partial^2 h}{\partial \eta^2} = v. \tag{23}$$

This equation is written in terms of the second-order derivative collocation matrix as

$$\sum_{j=1}^{N+2} B_{ij} h_j = v \tag{24}$$

where the fourth-order derivative of the h can be defined as

$$\frac{\partial^4 h}{\partial \eta^4} = \frac{\partial^2 v}{\partial \eta^2} = \sum_{j=1}^{N+2} \sum_{k=1}^{N+1} B_{ik} B_{kj} h_j + B_{i,1} v_1 + B_{i,N+2} v_{N+2}$$

or

$$\frac{\partial^4 h}{\partial \eta^4} = \frac{\partial^2 v}{\partial \eta^2} = \sum_{j=1}^{N+2} BB_{ij} h_j + B_{i,1} v_1 + B_{i,N+2} v_{N+2} \tag{25}$$

where

$$BB_{ij} = \sum_{k=1}^{N+1} B_{ik} B_{kj}.$$

The third-order derivative of h takes the form

$$\frac{\partial^3 h}{\partial \eta^3} = \frac{\partial^v}{\partial \eta} = \sum_{j=1}^{N+2} \sum_{k=2}^{N+1} A_{ik} B_{kj} h_j + A_{i,1} v_1 + A_{i,N+2} v_{N+2}$$

$$= \sum_{j=1}^{N+2} AB_{ij} h_j + A_{i,1} v_1 + A_{i,N+2} v_{N+2}$$

where

$$AB_{ij} = \sum_{k=2}^{N+1} A_{ik} B_{kj}, \quad i = 1, 2, \ldots, N+2. \tag{26}$$

We substitute;

$$\zeta = 4\eta^2 \tag{27}$$

to obtain;

$$h_\eta = 4\sqrt{\zeta} h_\zeta \tag{28}$$

and

$$h_{\eta\eta} = 16\zeta h_{\zeta\zeta} + 8h_\zeta. \tag{29}$$

Now we apply collocation in the new domain $\zeta \in [0, 1]$, corresponding to half the length while satisfying the differential equations at N interior points. At $\zeta = 1$, we have;

$$\frac{\partial h}{\partial \zeta} = \frac{\partial v}{\partial \zeta} = 0. \tag{30}$$

So we have $N + 2$ equations in $N + 2$ unknowns (h_i, $i = 1, 2, \ldots, N + 1$, v_{N+1}).

We also have symmetry around $\zeta = 1$, so we can use spline collocation at $\zeta = \lambda$, such that we have continuity of the function and its first, second and third derivative at the spline point $\zeta = \lambda$, and apply the above second-order approach on both sides of the spline point.

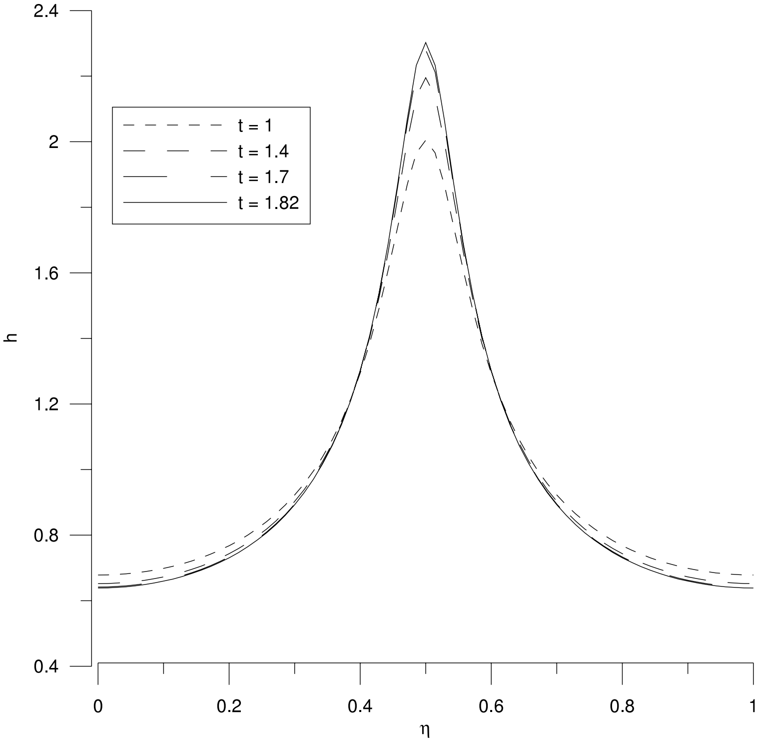

Fig. 2. Coating flow on a stationary cylinder with $A = -1.5$, $Bo = 1000\,000$.

4. Results and discussion

The case of a stationary cylinder ($W = 0$) is studied first.

For the case of a negative Hamaker constant ($A = -1.5$), a steady state is reached. This is shown in Fig. 2 for a spline collocation method with 14 splines and a tenth-order polynomial in each spline ($\alpha = \beta = 0$), and in Fig. 3 for a two spline second-order symmetric method for half the cylinder circumference with an 18th-order polynomial in each spline ($\alpha = 1$, $\beta = -0.5$) and a spline point at $\zeta = 0.934$. A negative Hamaker constant occurs when the dielectric constant of the solid is greater than that of the liquid [4]. In this case van der Waals forces are stabilizing.

For $A = 0.1$ and $Bo = 20$ (Figs. 4 and 5) van der Waals forces become destabilizing, rupture is expected and this happens at $t = 1.82$. Rupture is assumed to occur when the dimensionless film thickness is less than 0.05. In Fig. 5 we used a spline point at $\zeta = 0.61$. Reisfeld and Bankoff [4] obtained a rupture time of 1.6 using 64 spectral modes of a Fourier spectral method. This means that they need more spectral modes to get an accurate solution.

Next, we study the case of a rotating cylinder. Evans et al. [7] presented a simplified analysis which indicated that there is a critical rotation velocity ($Wc = 0.0141$) below and above which the film dynamics differs. For $0 < W < Wc$, a steady state is reached. The fluid is carried toward the upward-moving side of the cylinder with a maximum in the layer thickness close to an angle $\Theta = 0.75 * 2\pi$. A minimum exists close to this maximum and another at $\Theta = 0.25 * 2\pi$ where drainage and rotation occur in the direction. These phenomena are shown in Figs. 6, 8.

For the case of $W = 0.012$ and $Bo = 14\,444$ (Fig. 6) the approach to the steady state from a uniform initial condition is uniform with the maximum height occurring at an angle $\Theta = 0.607 * 2\pi$. For $Bo = 20$ (Fig. 8) the maximum overshoots initially, but occurs at a fixed angle of about $\Theta = 0.714 * 2\pi$.

For the case of $W = 0.016 > Wc$, $Bo = 14\,444$ (Fig. 7) and $Bo = 20$ (Fig. 9), the approach to steady state is no longer smooth, and a front of liquid forms. This front is carried out in the direction of rotation. The front gradually decreases in size. Four hundred grid points of a finite difference scheme are used by Evans et al. [7] to simulate the cases of $Bo = 14\,444$ whereas we are using 14 splines with a tenth-order polynomial in each spline making a total of 140 points.

To the best of our knowledge the effect of the presence of van der Waals forces on a rotating cylinder was not studied before. The effect of van der Waals forces is shown in Figs. 10–13. For small Hamaker constant ($A = 0.1$) (Figs. 10 and 11) little effect is shown in comparison with Figs. 8 and 9. This means that rotation can prevent rupture. For $A = 0.5$, rupture occurs (Figs. 12 and 13).

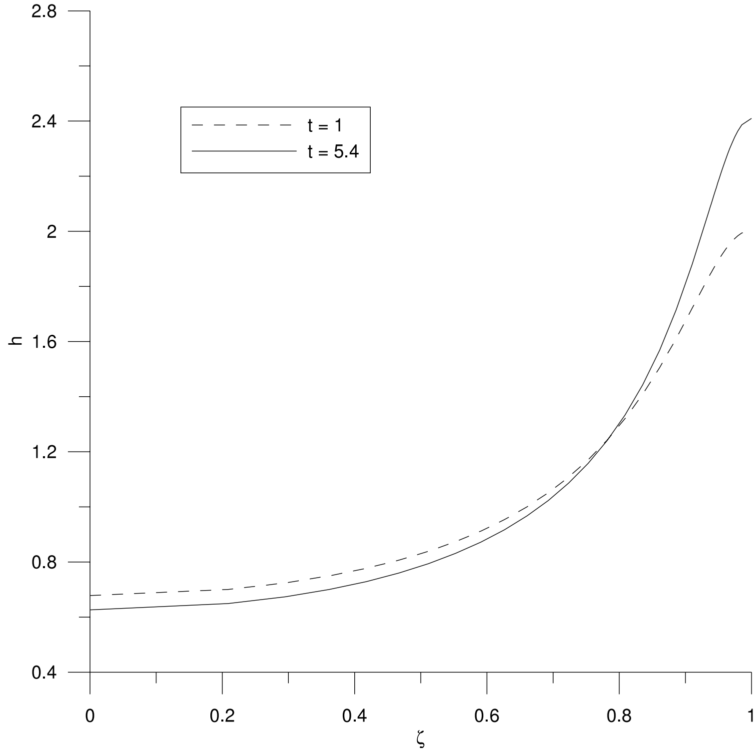

Fig. 3. Coating flow on a stationary cylinder with $A = -1.5$.

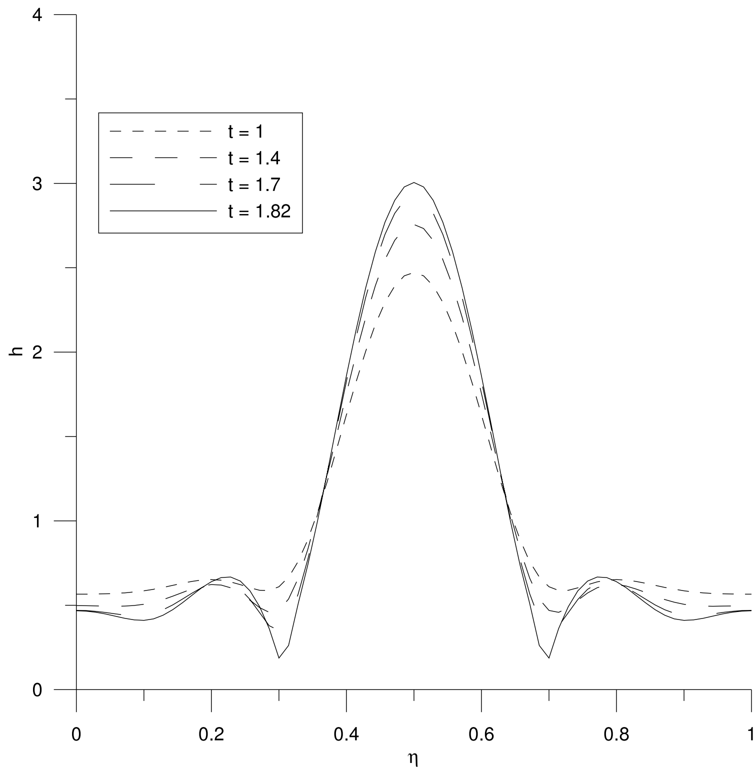

Fig. 4. Coating flow on a stationary cylinder with $A = 0.1$, $Bo = 20$.

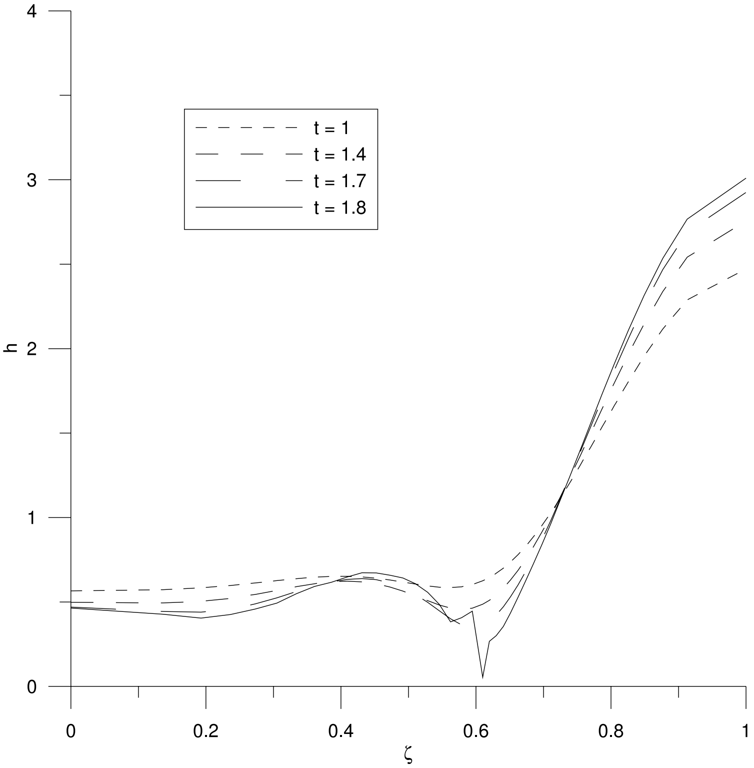

Fig. 5. Coating flow on a stationary cylinder with $A = 0.1$, $Bo = 20$.

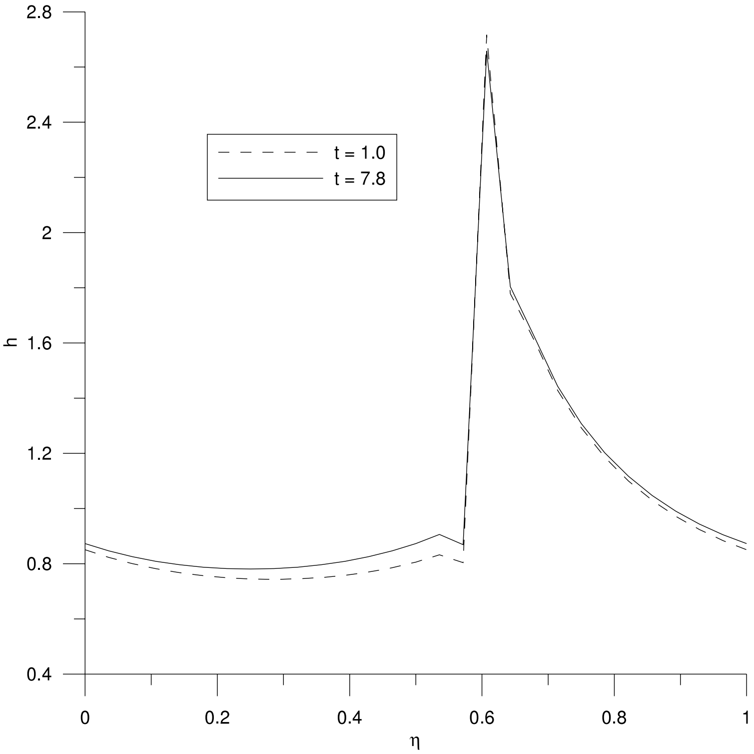

Fig. 6. Coating flow on a rotating cylinder with $W = 0.012$, $Bo = 14\,444$.

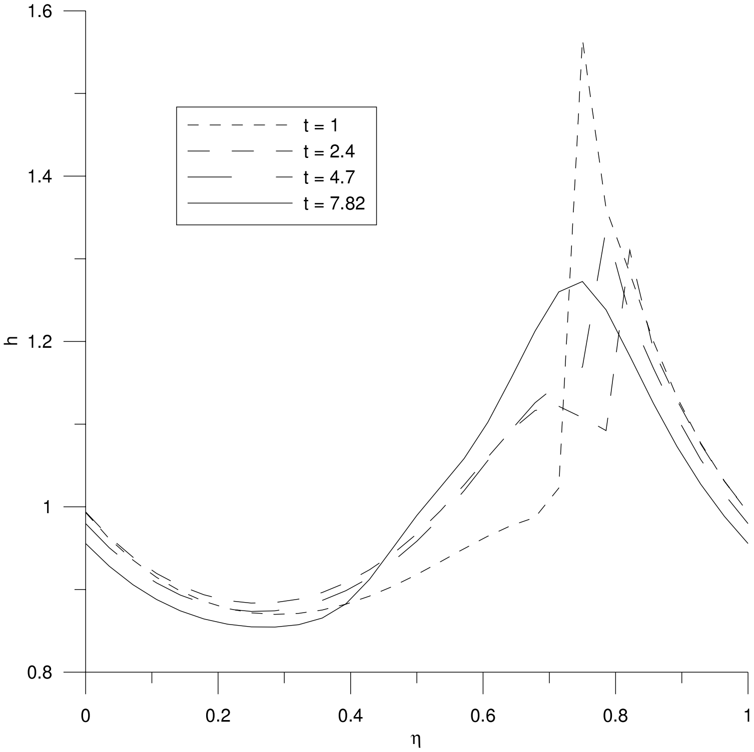

Fig. 7. Coating flow on a rotating cylinder with $W = 0.016$, $Bo = 14\,444$.

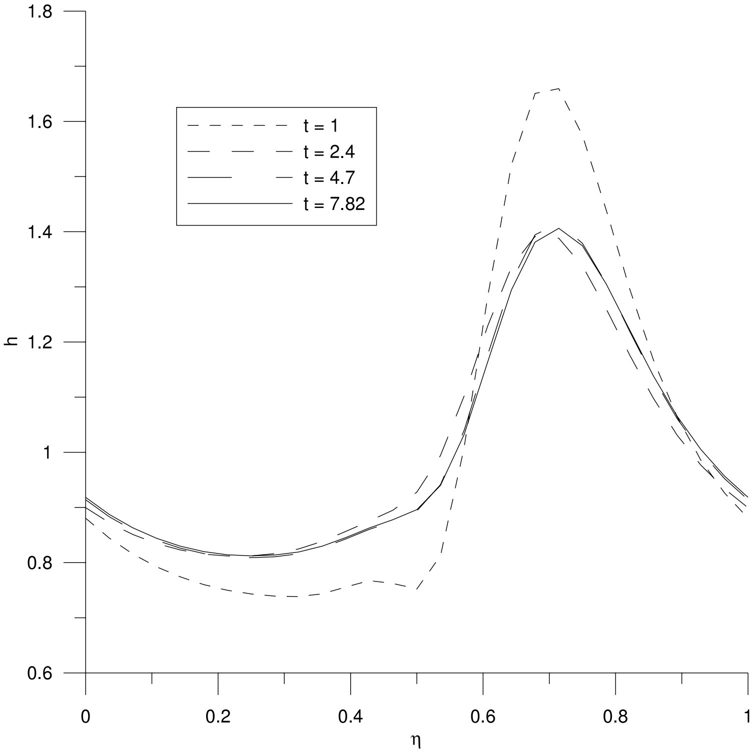

Fig. 8. Coating flow on a rotating cylinder with $W = 0.012$, $Bo = 20$.

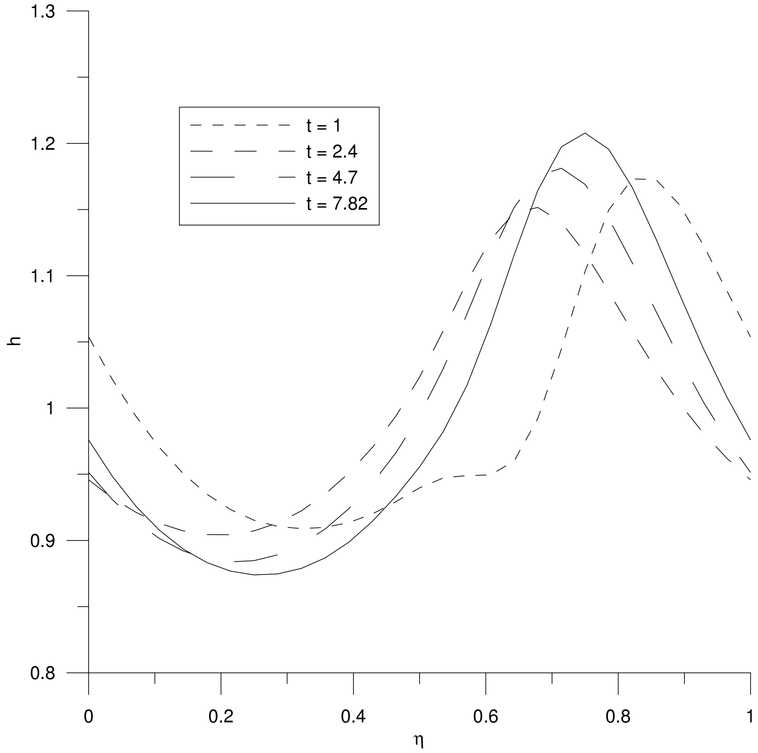

Fig. 9. Coating flow on a rotating cylinder with $W = 0.016$, $Bo = 20$.

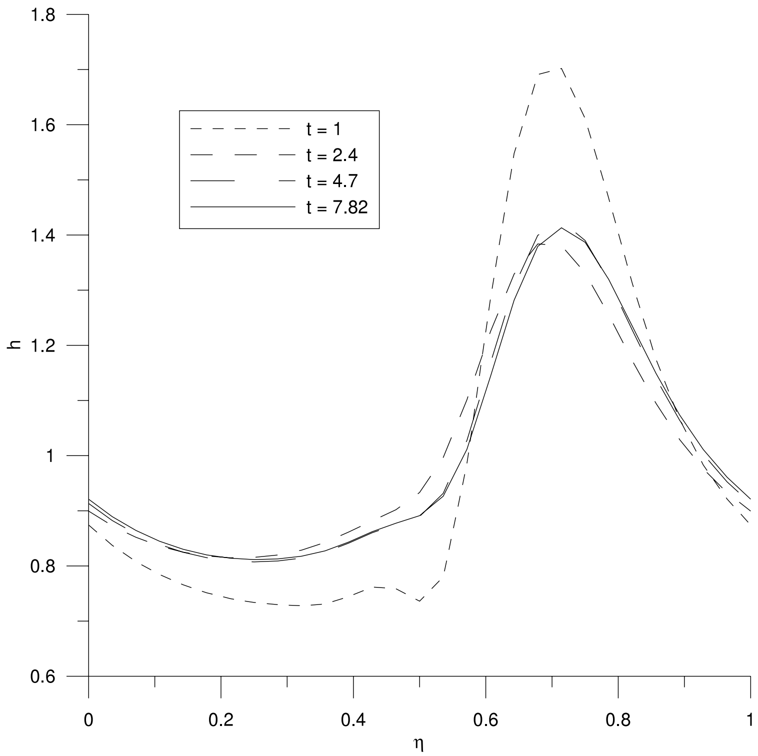

Fig. 10. Coating flow on a rotating cylinder with $A = 0.1$, $W = 0.012$, $Bo = 20$.

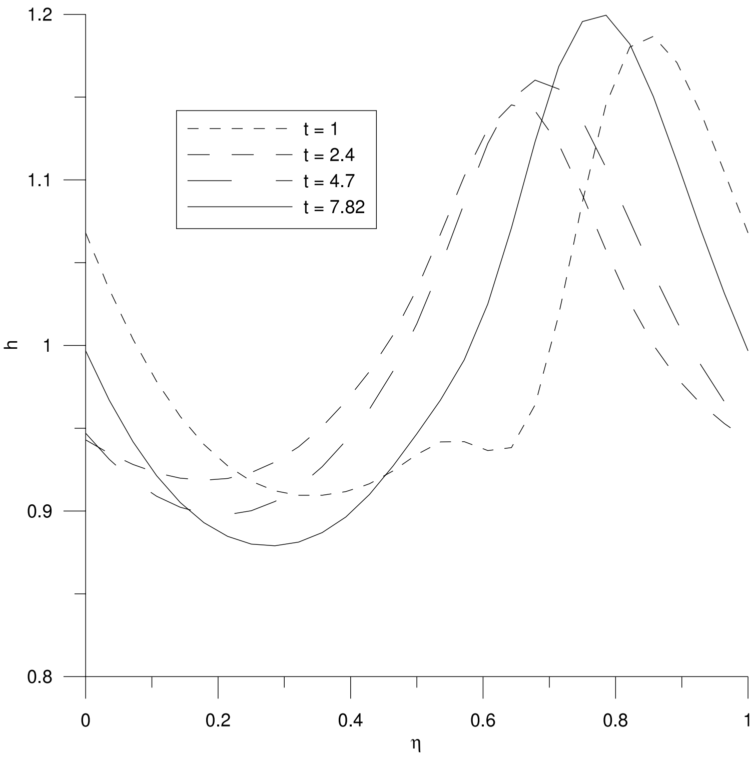

Fig. 11. Coating flow on a rotating cylinder with $A = 0.1$, $W = 0.016$, $Bo = 20$.

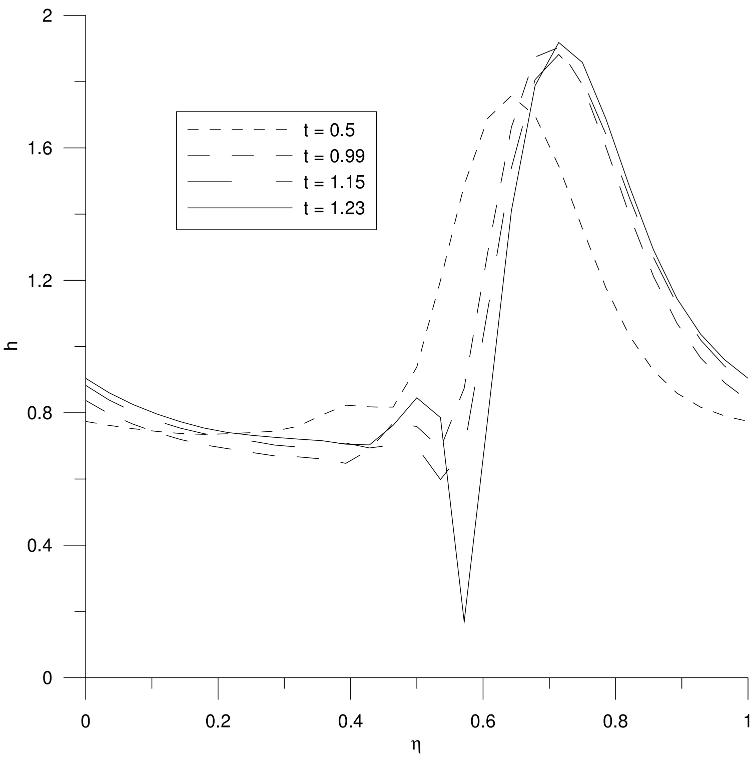

Fig. 12. Coating flow on a rotating cylinder with $A = 0.5$, $W = 0.012$, $Bo = 20$.

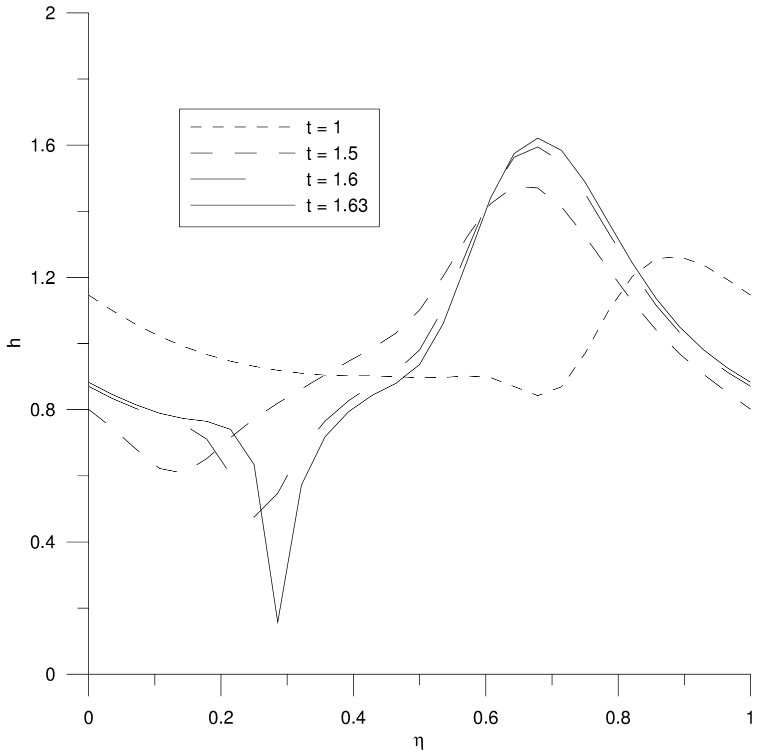

Fig. 13. Coating flow on a rotating cylinder with $A = 0.5$, $W = 0.016$, $Bo = 20$.

5. Conclusions

The present work examines the dynamics of coating films of liquids flowing on a horizontal cylinder taking into consideration intermolecular van der Waals forces which become active as the film gets thinner. Increasing the rotation speed makes the film more uniform. Below a critical speed determined by Evans et al. [7] to be $W = 0.0141$, the coating profile reaches steady state uniformly and quickly. Above this speed, a front is formed and this front decreases in size with time and a steady state is reached after a long time. For small van der Waals forces rotation may prevent rupture. The numerical solution of the evolution equation is carried out using efficient spline collocation methods. A detailed comparison with other numerical methods used for thin film flow methods needs to be carried out.

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