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Approximate Analytical Solution for the Isothermal Lane-Emden Equation in a Spherical Geometry

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APPROXIMATE ANALYTICAL SOLUTION FOR THE ISOTHERMAL LANE EMDEN EQUATION IN A SPHERICAL GEOMETRY

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RESUMEN

Este trabajo obtiene una solución analítica aproximada para la ecuación isoterma de Lane-Emden que modela una esfera isotérmica autogravitante. La solución aproximada se obtiene en términos de parámetros de distancias pequeños y grandes por el método de perturbaciones. La solución aproximada se compara con la solución númerica. La solución aproximada obtenida es válida para todos los valores del parámetro de distancia.

ABSTRACT

This paper obtains an approximate analytical solution for the isothermal Lane-Emden equation that models a self-gravitating isothermal sphere. The approximate solution is obtained by perturbation methods in terms of small and large distance parameters. The approximate solution is compared with the numerical solution. The approximate solution obtained is valid for all values of the distance parameter.

Key Words: stars: formation — stars: general

1. INTRODUCTION

Several theoretical studies and numerical methods have been focused on the study of the isothermal Lane Emden equation or Frank-Kamenetskii equation (Adler 2011, Frank-Kamenetskii 1955, Aris 1975, Britz Strutwolf and Osterby 2011, Chandrasekhar 1967, Enig 1967, Hlavacek and Marek 1968, Steggerda 1965, Moise and Pritchard 1989, Nazari-Golshan, Nourazar, Ghafoori-Fard, Yildirim and Campo 2013, Mirza 2009, Liu 1996). The equation models a thermal explosion in a closed enclosure. The equation for a spherical enclosure comes from the theory of self gravitating molecular cores; it models an isothermal gas sphere. The equation also models a non-isothermal zero order reaction in a catalyst particle. Frank-Kamenetskii (1955) formulated the problem and obtained analytical solutions for the case of a slab and a cylinder enclosure. Other results for the case of a slab and a cylinder can be found in the references (Boyd 2011, Soliman 2013, Harley and Momoniat 2008). Solutions for the case of a spherical enclosure case can so far only be obtained numerically.

The isothermal Lane Emden equation is given by

$$\frac{d^2\psi}{d\xi^2} + \frac{s}{\xi}\frac{d\psi}{d\xi} = \exp(-\psi) \tag{1}$$

$$\psi(0) = 0 \tag{2}$$

$$\left.\frac{d\psi}{d\xi}\right|_{\xi=0} = 0 \tag{3}$$

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where s is a shape factor that takes a value of 0 for an infinite slab, 1 for an infinite cylinder and 2 for a sphere.

The initial value problem given by equations (1-3) models a self-gravitating isothermal sphere.

In this form the problem is called Lane-Emden equation of the second kind or isothermal Lane-Emden equation. Many researchers have obtained an approximate analytical solution for a small ξ. These efforts are summarized in the work of Iacono and De Felice (2014) who gave a better approximation. Other approximations were derived by Liu (1996), and Mirza (2009). Furthermore Raga et al. (2013) obtained two approximate solutions for small and large ξ. In this paper we obtained in a single expression an approximate solution valid for all ξ. Preliminary results of the present work were presented in a conference (Soliman 2014). Tables of ψ versus ξ for the case of a sphere are taken from Chandrasekhar and Wares (1949) and Horedt (1986).

The aim of the present paper is to derive an approximate analytical solution for equations (1-3).

The paper is organized as follows: In the next section, we obtain approximate solutions for the initial value problem for large and small distance parameters using perturbation methods. Then we modify the solution obtained for large distance parameters to be also valid for the case of small distance parameters. Thereby we obtain a general approximate solution valid for all values of the distance parameter. Finally numerical results are presented.

2. MATHEMATICAL DEVELOPMENT

2.1. *Approximate solution for large ξ*

We consider equation (1) for the case of a sphere (s=2).

We notice that

$$\psi = 2\ln(\xi) - \ln(2) \tag{4}$$

satisfies equation (1) but does not satisfy initial conditions (2, 3). This solution is approached as ξ tends to infinity. It is a singular solution (Aris 1975). A non-singular solution cannot be obtained analytically. To obtain an approximate non-singular soltion, we proceed as follows:

Define:

$$\zeta = \ln(\xi)\,, \tag{5}$$

$$\phi = \psi - 2\ln(\xi) + \ln(2) = \psi - 2\zeta + \ln(2)\,. \tag{6}$$

Substituting equation (6) into equation (1), we obtain

$$\frac{d^2\varphi}{d\zeta^2} + \frac{d\varphi}{d\zeta} + 2 = 2\exp(-\varphi)\,. \tag{7}$$

For small φ (this happens as ξ tends to ∞), we can approximate equation (7) by

$$\frac{d^2\varphi}{d\zeta^2} + \frac{d\varphi}{d\zeta} + 2\varphi = 0\,, \tag{8}$$

which has a solution in the form

$$\varphi = A\exp(-\frac{\zeta}{2})\sin(B + \frac{\sqrt{7}}{2}\zeta)\,. \tag{9}$$

Or

$$\varphi = \frac{A}{\sqrt{\xi}}\sin(B + \frac{\sqrt{7}}{2}\ln\xi)\,, \tag{10}$$

where A and B are constants to be determined from the numerical solution. Since φ is small, we can write:

$$\varphi = \ln\left[1 + \frac{A}{\sqrt{\xi}}\sin(B + \frac{\sqrt{7}}{2}\ln\xi)\right]\,. \tag{11}$$

This formula is similar to that obtained by Chandrasekhar (1967) and Adler (2011).

Thus from equations (6) and (11), we have

$$\psi = \ln\left[\frac{\xi^2}{2}\left(1+\frac{A}{\sqrt{\xi}}\sin(B+\frac{\sqrt{7}}{2}\ln\xi)\right)\right] . \tag{12}$$

We solved equations (1-3) numerically using the DASSL routine (Petzold 1982) and determined the constants A and B. We used the two points (ξ, ψ)=(210650.464, 23.823) and (525665.99, 25.653) to obtain A=−1.178 and B=−0.507787. The first value corresponds to a point at which $\phi = 0$, while the second value corresponds to a point at which,

$$\frac{d\psi}{d\zeta} = 2 . \tag{13}$$

Equation (12) becomes

$$\psi = \ln\left[\frac{\xi^2}{2}\left(1-\frac{1.178}{\sqrt{\xi}}\sin(-0.507787+\frac{\sqrt{7}}{2}\ln\xi)\right)\right] . \tag{14}$$

We could extend the range of applicability of equation (14) by writing

$$\psi = \ln\left[1+\frac{\xi^2}{2}\left(1-\frac{1.178}{\sqrt[4]{2+\xi^2}}\sin(-0.507787+\frac{\sqrt{7}}{4}\ln(2+\xi^2))\right)\right] , \tag{15}$$

since this formula satisfies the initial conditions (2) & (3).

2.2. *Approximate solution for small* ξ

For small ξ, and ψ, equation (1) can be approximated by

$$\frac{d^2\psi}{d\xi^2} + \frac{s}{\xi}\frac{d\psi}{d\xi} = 1 - \psi , \tag{16}$$

whose solution (for $s = 2$) is given by

$$\psi = 1 - \sin(\xi)/\xi . \tag{17}$$

This approximation was suggested by Moise and Pritchard (1989).

The following Padé approximant was obtained by Nazari-Golshan et al. (2013) using the homotopy perturbation method combined with a Fourier transform

$$\psi = \frac{\frac{\xi^2}{6} + \frac{15587041\xi^4}{319406280} + \frac{72143758\xi^6}{27668569005}}{1 + \frac{912438\xi^2}{2661719} + \frac{78025283\xi^4}{2635101810} + \frac{3508303919\xi^6}{5691819909600}} . \tag{18}$$

Iacono and De Felice (2014) presented many new Padé approximations. The best of them is given by:

$$\psi = \ln\left[1+\frac{\xi^2}{6}+\frac{\xi^4}{180}\frac{(1+4\xi^2/189)}{(1+\xi^2/27+\xi^4/4115)}\right] . \tag{19}$$

We can add to this the following approximation which will approach the singular solution as ξ tends to infinity

$$\psi = \ln\left[1+\frac{\xi^2}{6}+\frac{\xi^4}{180}\frac{(1+271\xi^2/9240)}{(1+179\xi^2/3960+271\xi^4/(60*9240))}\right] . \tag{20}$$

We derive here another approximation. For this purpose we use the following transformation:

$$\psi = 2\ln(z) ; \tag{21}$$

thus:

$$\frac{d\psi}{d\xi} = \frac{2}{z}\frac{dz}{d\xi}\,, \tag{22}$$

$$\frac{d^2\psi}{d\xi^2} = \frac{2}{z}\frac{d^2z}{d\xi^2} - \frac{2}{z^2}\left(\frac{dz}{d\xi}\right)^2. \tag{23}$$

Substituting equations (21-23) into equation (1), we obtain:

$$\frac{2}{z}\frac{d^2z}{d\xi^2} - \frac{2}{z^2}\left(\frac{dz}{d\xi}\right)^2 + \frac{2s}{\xi \cdot z}\frac{dz}{d\xi} = \frac{1}{z^2}. \tag{24}$$

Multiplying both sides of equation (1) by $(2\frac{d\psi}{d\xi})$ and integrating with respect to ξ, we obtain:

$$\left(\frac{d\psi}{d\xi}\right)^2 = -2(\exp(-\psi) - 1) - 2s\int_0^\xi \frac{1}{\xi}\left(\frac{d\psi}{d\xi}\right)^2 d\xi = \frac{4}{z^2}\left(\frac{dz}{d\xi}\right)^2. \tag{25}$$

Substituting equations (21) into equation (25), we obtain:

$$\frac{4}{z^2}\left(\frac{dz}{d\xi}\right)^2 = -2(\frac{1}{z^2} - 1) - 8s\int_0^\xi \frac{1}{\xi z^2}\left(\frac{dz}{d\xi}\right)^2 d\xi. \tag{26}$$

Substituting equation (26) into equation (24) we obtain:

$$\frac{d^2z}{d\xi^2} + \frac{s}{\xi}\frac{dz}{d\xi} = \frac{1}{2}z - 2sz\int_0^\xi \frac{1}{\xi z^2}\left(\frac{dz}{d\xi}\right)^2 d\xi. \tag{27}$$

For the case of a cylinder ($s = 1$), equation (1) can be solved analytically to give:

$$\psi = 2\ln(1 + \xi^2/8)\,. \tag{28}$$

And

$$z = 1 + \xi^2/8\,, \tag{29}$$

$$\frac{1}{\xi}(\frac{dz}{d\xi}) = \frac{1}{4}\,, \tag{30}$$

$$\int_0^\xi \frac{1}{\xi z^2}\left(\frac{dz}{d\xi}\right)^2 d\xi = \int_0^\xi \frac{1}{4z^2}(\frac{dz}{d\xi})d\xi = \frac{1}{4}(1 - \frac{1}{z}). \tag{31}$$

From equations (31), equation (27) becomes:

$$\frac{d^2z}{d\xi^2} + +\frac{1}{\xi}\frac{dz}{d\xi} = \frac{1}{2}\,. \tag{32}$$

The main idea for the derivation of the approximate solution for the case of a sphere ($s = 2$) is to assume that:

$$8\int_0^\xi \frac{1}{\xi z^2}\left(\frac{dz}{d\xi}\right)^2 d\xi \cong C(1 - \frac{1}{z})\,, \tag{33}$$

which has a similar form to equation (31).

For $s = 2$, a first order expansion of equation (1) around the solution ψ for small ξ would give:

$$\psi \cong \frac{\xi^2}{6}\,. \tag{34}$$

Thus,

$$z = \exp(\psi/2) \cong 1 + \frac{\xi^2}{12}. \tag{35}$$

Substituting equations (35) into equation (33) we obtain

$$\frac{\xi^2}{9} = C(\frac{\xi^2}{12}). \tag{36}$$

This gives a first approximation of C as:

$$C = \frac{4}{3}. \tag{37}$$

From equation (33), equation (27) becomes in the case of a sphere:

$$\frac{d^2 z}{d\xi^2} + \frac{2}{\xi}\frac{dz}{d\xi} = \frac{C}{2} - \frac{(C-1)z}{2}. \tag{38}$$

Equation (38) can be solved to give:

$$\psi = 2\ln(z) = 2\ln\left[1 + \frac{1}{(C-1)}(1 - \frac{\sin(c\xi)}{\xi(c)})\right]; \tag{39}$$

$$c = \sqrt{\frac{(C-1)}{2}}. \tag{40}$$

We can improve this formula to make it satisfy the condition that the singular solution is approached as ξ tends to infinity, obtaining:

$$\psi = \ln\left[1 + \frac{\xi^2}{2}(1 - \frac{2}{3}\frac{\sin(c_1\xi)}{\xi(c_1)})\right]. \tag{41}$$

Choosing $c_1 = \sqrt{0.1}$ will make the expansion of (41) satisfy equation (43).

2.3. *Approximate solution for all ξ*

The following formula was derived from equation (15)

$$\psi = \ln\left[1 + \frac{\xi^2}{2}\left(1 - \frac{2}{3.\sqrt[4]{1+\xi^2/15}}\cos(\frac{\sqrt{7}}{4}\ln(1+\xi^2/23.162231))\right)\right]. \tag{42}$$

Notice that

$$\exp[\frac{4}{\sqrt{7}}(0.507787 + \frac{\pi}{2})] = 23.162231 \quad .$$

Equation (42) contains the main components obtained for a large ψ, and satisfies the approximation for small ψ:

$$\psi \cong \frac{1}{6}\xi^2 - \frac{1}{120}\xi^4. \tag{43}$$

For equation (42) to approach equation (15) with large ξ, the constant that pre-multiplies the cosine function (2/3) should be 0.59858, since

$$1.178/\sqrt[4]{15} = 0.59858 \quad .$$

We can thus propose a general formula of the following form

$$\psi = \ln\left[\ 1 + \frac{\xi^2}{2}\left(1 - \frac{(2/3 + 0.59858 * 10^{-6}\xi^6)}{(1+10^{-6}\xi^6)\sqrt[4]{1+\xi^2/15}}\cos(\frac{\sqrt{7}}{4}\ln(1+\xi^2/23.162231))\right)\right]. \tag{44}$$

Raga et al. (2013) obtained the following formula (after modifying the distance parameter),

$$\psi = \ln\left[\frac{\xi^2}{2} / \left(1 + \frac{1.8004}{\sqrt{\xi}} \cos(\frac{\sqrt{7}}{2} \ln \xi + 4.2111)\right)\right] \tag{45}$$

which is valid for $\xi > 67.711$. For $\xi < 67.711$, they used an approximation derived by Hunter (2001).

They were also able to extend the applicability of equation (45) to a lower value of ξ by making the equation parameters to depend on ξ.

3. NUMERICAL RESULTS AND DISCUSSION

We chose the DASSL FORTRAN code (Petzold 1982) to solve equations (1-3). This code uses Backward Differentiation Formula methods to solve a system of differential algebraic equations. We were able to reproduce the table of Horedt (1986) up to a value of ψ=14.7 .

The numerical solution allowed us to plot ψ versus ξ (Figure 1). Equation (44) is slightly better than equation (42) for the numerically exact approximation and for other approximations, but both of them are indistinguishable from the numerical solution in Figure 1. In addition, equations (19),(20),(41) and (45) are indistinguishable from the numerical solution. Equation (17) is accurate up to $\xi = 2.4$. Equation (18) is accurate up to $\xi = 3.8$. Equation (39) is accurate up to $\xi = 3.4$.

The graphs of Figures 2 and 3 were drawn to differentiate between the accuracy of equations (19),(20),(41),(42),(44) and (45). Figure 2 shows that equation (19) is very accurate up to about ξ=30. Equations (20) and (22) lose accuracy in the intermediate region but regain it at large values of ξ because they include the singular solution. Figure 3 shows that equation (44) is the best for all ξ, followed by equation (42) and (45). The accuracy of equations (42) and (44) is less than 0.4%.

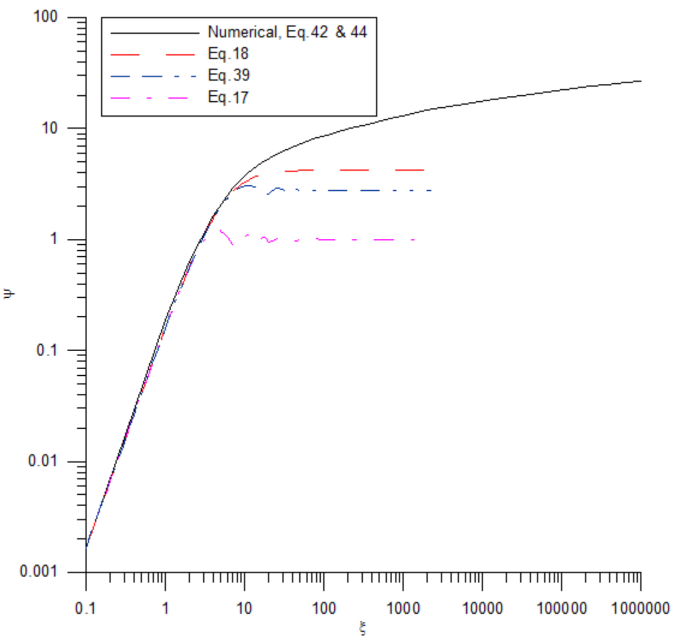

Fig. 1. ψ versus ξ for the numerical solution and for the approximations given by equations (17), (18), (39), (42) and (44).

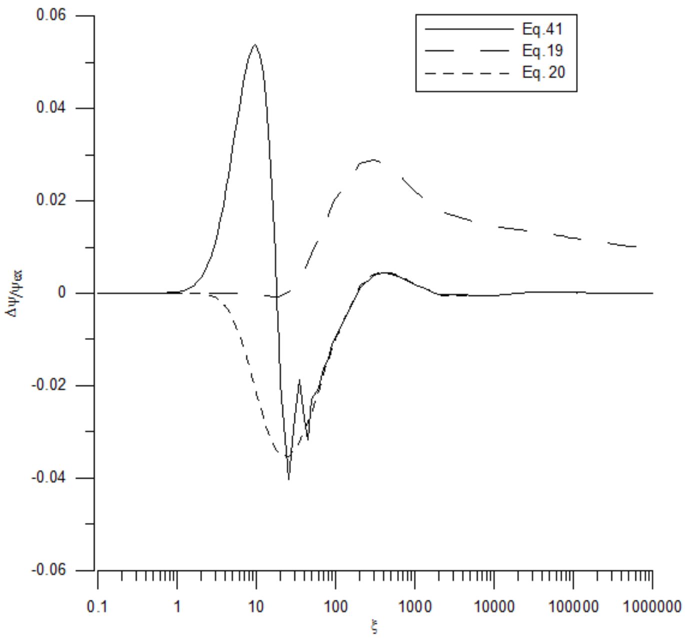

Fig. 2. Relative error in ψ for approximations given by equations (19), (20) and (41).

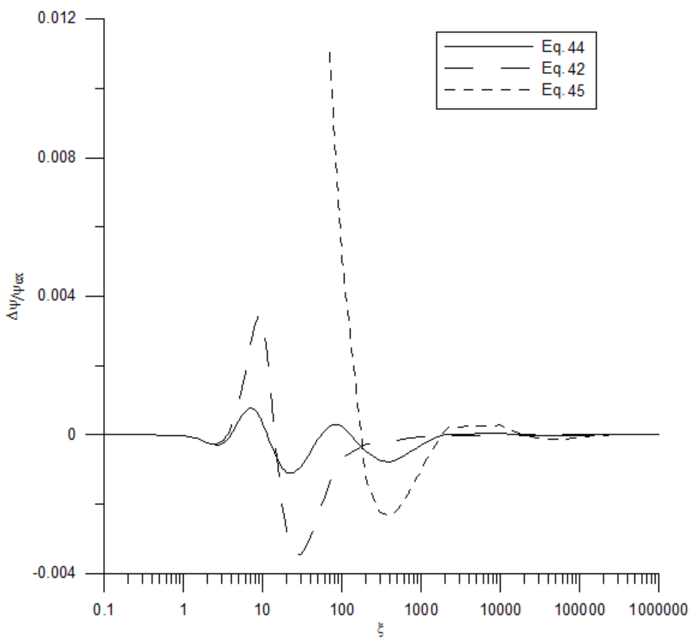

Fig. 3. Relative error in ψ for approximations given by equations (42), (44) and (45).

4. CONCLUSIONS

We derived an approximate analytical solution for the isothermal Lane-Emden equation that is valid for all values of ξ. Equations (42) and (44) are highly accurate and better than any equation available in the literature. Equation (41), which is very simple, has a good accuracy except for the small intermediate region. It could be improved further if its expansion was made to correspond with larger values than those used in equation (43). An extension to the case of a hollow sphere is currently under investigation.

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