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Approximate solution for the Lane-Emden equation of the second kind in a spherical annulus

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ABSTRACT

In this paper, we derive accurate approximate solution of Lane-Emden equation of the second kind in a spherical annulus geometry. The approximate solution is obtained by analytic arguments, and perturbation methods in terms of small and large radial distance parameter. The approximate solution is compared with the numerical solution. The approximate solution obtained is valid for all values of the radial distance parameter.

Our best approximation has a maximum relative error in the dependent variable of 20%. In most cases it is much less than this value. This maximum error decreases as the radius of the annulus increases.

1. Introduction

Several theoretical studies and numerical methods were used for the study of the initial value problem of the isothermal Lane-Emden equation or the related boundary value problem of the Frank-Kamenetskii equation (Adler, 2011; Aris, 1975;Bazley and Wake, 1981,Britz et al., 2011; Chandrasekhar, 1967; Enig, 1967; Frank-Kamenetskii, 1955; Gustafson and Eaton, 1982; Hlavacek and Marek, 1968; Moise and Pritchard, 1989; Nazari-Golshan et al., 2013; Steggerda, 1965). The Lane-Emden equation is a basic equation in astrophysics that describes dimensionless density distribution in an isothermal gas sphere, temperature variation of a self-gravitating star, and thermal behavior of a spherical cloud of a gas under mutual attraction of its molecules. On the other hand, thermal explosion in an enclosure and non-isothermal zero order reaction in a catalytic pellet are modeled as boundary value problem for which analytical solutions for the case of a slab and cylinder enclosure were obtained by Frank-Kamenetskii (1955).

More results on the case of a slab and cylinder can be found in references(Boyd, 2011; Christodoulou and Kazanas, 2008; Harley and Momoniat, 2008; Reger and Van Gorder, 2013; Soliman, 2013). A closed form analytical solution for the spherical enclosure is not available. Soliman and Al-Zeghayer (2015) obtained approximate analytical solution for the case of solid sphere. More results in the case of solid sphere can be found in references (Iacono and De Felice, 2014; Nouh, 2004; Raga et al.,2013; Van Gorder, 2011).

Many numerical methods can be used to solve such a highly non-linear problems, e.g., collocation methods (Soliman and Alhumaizi, 2008, 2005; Alhumaizi and Soliman, 2000), and Pade approximants (Khader., 2013). However, in this paper we are interested in obtaining approximate analytical solution to the case of a spherical annulus.

The initial value problem of Lane-Emden equation of the second kind can be written as

$$\frac{d^2\psi}{d\xi^2} + \frac{s}{\xi}\frac{d\psi}{d\xi} = \exp(-\psi) \quad (1)$$

subject to the initial conditions

$$\psi(\xi_0) = 0 \quad (2)$$

$$\left.\frac{d\psi}{d\xi}\right|_{\xi=\xi_0} = 0 \quad (3)$$

where ψ is the dependent variable, ξ is the radial distance variable, ξ_0 is the annulus radial distance and s = 0 for a slab, s = 1 for a cylinder and s = 2 for a sphere.

The analytical solution in the case of slab (s = 0) is given by Hlavacek and Marek (1968),

$$\psi_{sl} = 2\ln\left(\cosh\left(\frac{\xi - \xi_0}{\sqrt{2}}\right)\right) \quad (4)$$

The analytical solution in the case of infinite cylinder (s = 1) is given by Hlavacek and Marek (1968), and Christodoulou and Kazanas (2008),

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$$\psi_{cl} = 2\ln\left(\frac{(n+1)+(n-1)\left(\frac{\xi}{\xi_0}\right)^{2n}}{2n\left(\frac{\xi}{\xi_0}\right)^{n-1}}\right) \tag{5}$$

where (n) is given by

$$n = \sqrt{1+\frac{\xi_0^2}{2}} \tag{6}$$

For the case of $\xi_0 = 0$ Eq. (5) reduces to

$$\psi_{cl} = 2\ln\left(1+\frac{\xi^2}{8}\right) \tag{7}$$

The aim of the present paper is to give an approximate analytical solution for Eqs. (1)–(3) for the case of spherical annulus.

In the next section, we obtain approximate solution for this initial value problem for large and small distance parameter using perturbation methods. Then we modify the solution obtained for large distance parameter to be also valid for the small distance parameter case. Approximate solutions are also obtained for large radius of the spherical annulus. Then we obtain one general approximate solution valid for all values of the distance parameter and the radius of the spherical annulus. Finally numerical results are presented.

2. Mathematical development

2.1. Perturbation solution for small $(\xi - \xi_0)$

We first obtain the perturbation solution of Eqs. (1)–(3) around $\xi = \xi_0$.

For this purpose we define y as

$$y = \xi - \xi_0 \tag{8}$$

and λ as

$$\lambda = \frac{\xi_0}{\xi} \tag{9}$$

We obtain for small y and λ

$$\psi \cong \frac{1}{6}y^2(1+2\lambda) - \frac{1}{120}y^4(1+4\lambda) + \frac{1}{1890}y^6(1+6\lambda) + \frac{\lambda^2}{1080}\left(3\xi y^5 - 5\lambda\xi^2y^4 + 10\lambda^2\xi^3y^3 - 30\lambda^3\xi^4y^2 + 60\lambda^4\xi^5(\xi\ln(\xi)-\xi)\right) + \ldots \tag{10}$$

This can be further simplified for small λ to;

$$\psi \cong \frac{1}{6}y^2(1+2\lambda) - \frac{1}{120}y^4(1+4\lambda) + \frac{1}{1890}y^6(1+6\lambda) + \frac{\lambda^2}{1080}(3y^6 - 2\lambda y^6) \tag{11}$$

2.2. Asymptotic values for large ξ

It was shown in by Soliman and Al-Zeghayer (2015) that for large ξ, ψ can be approximated by

$$\psi = \ln\left(\frac{\xi^2}{2}\left(1+\frac{A}{\sqrt{\xi}}\cos\left(\frac{\sqrt{7}}{2}\ln(\xi/B)\right)\right)\right) \tag{12}$$

This solution does not satisfy the initial conditions (2,3). This solution can be modified to satisfy the initial conditions and satisfy the perturbation solution (11) to give

$$\psi = \ln\left(1+\frac{y^2}{2}\left(1-\frac{2}{3.\sqrt[4]{1+y^2/15}}\cos\left(\frac{\sqrt{7}}{4}\ln(1+y^2/\mathrm{as})\right)\right)\right) + \lambda\ln\left(1+y^2\left(1-\frac{2}{3.\sqrt[4]{1+2y^2/15}}\cos\left(\frac{\sqrt{7}}{4}\ln\left(1+y^2/\mathrm{az}\right)\right)\right)\right) \tag{13}$$

where

$$\mathrm{as} = \sqrt{(63*105/13)} \tag{14}$$

$$\mathrm{az} = \sqrt{(63*105/4)} \tag{15}$$

2.3. Asymptotic value for large ξ_0

For large ξ_0, the solution approaches that of a slab. To obtain this solution we proceed as follows

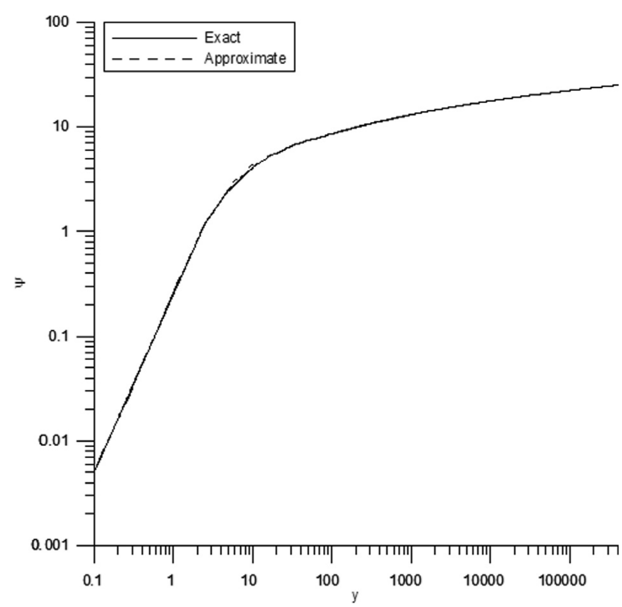

Fig. 1. Ψ against y for $\xi_0 = 1$

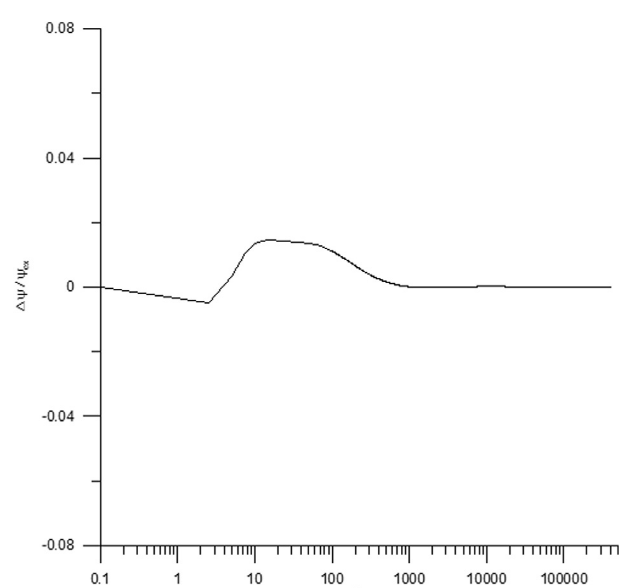

Fig. 2. Relative error in Ψ against y for the approximation for $\xi_0 = 1$

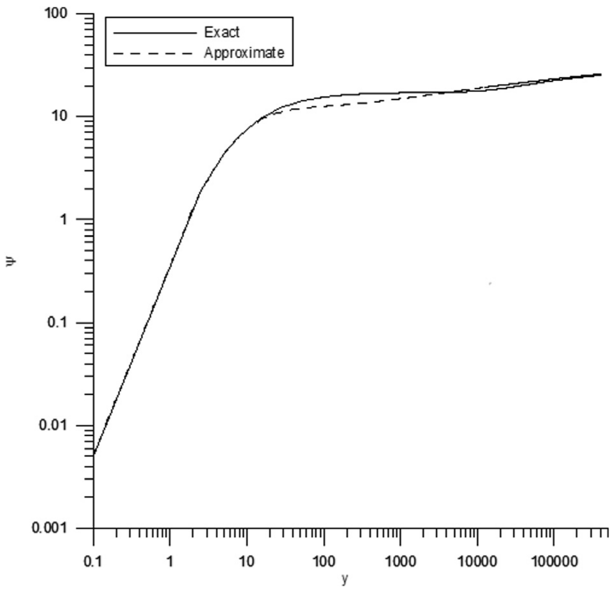

Fig. 3. Ψ against y for $\xi_0 = 10$

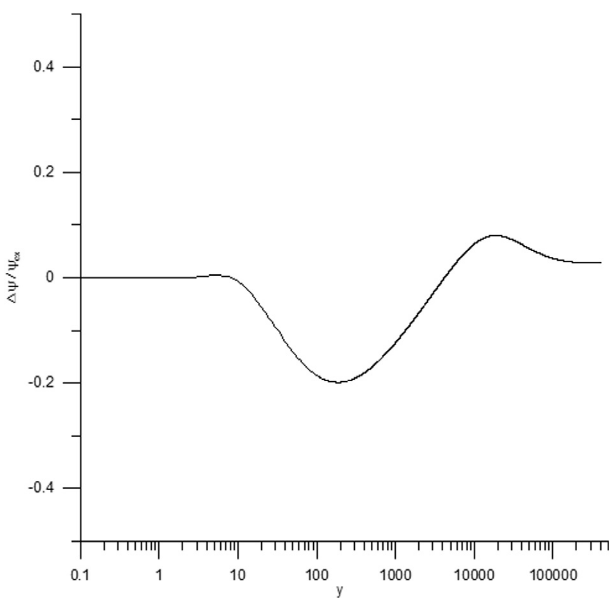

Fig. 4. Relative error in Ψ against y for the approximation for $\xi_0 = 10$

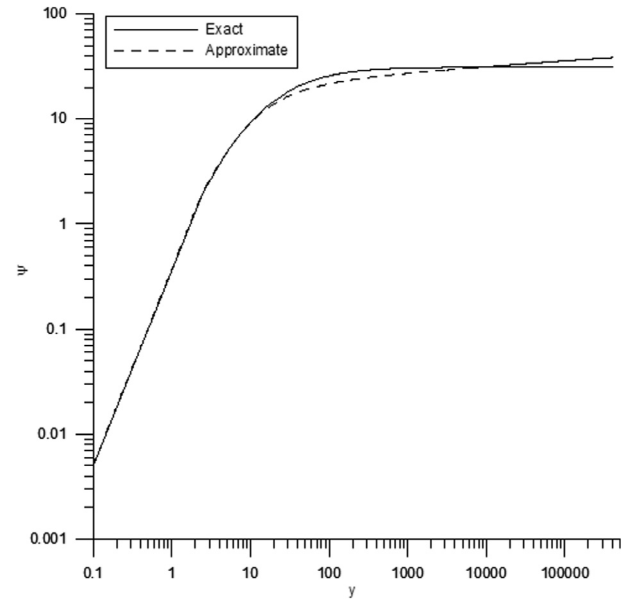

Fig. 5. Ψ against y for $\xi_0 = 20$

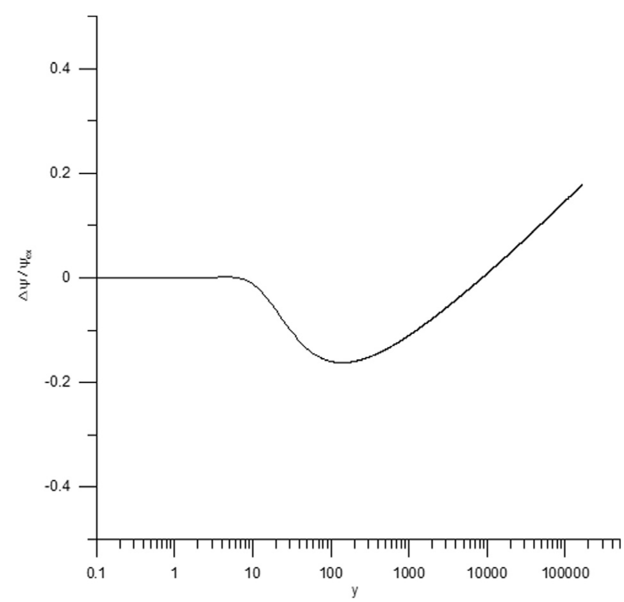

Fig. 6. Relative error in Ψ against y for the approximation for $\xi_0 = 20$

Let

$$\eta = \frac{1}{\xi} \text{ and } \eta_0 = \frac{1}{\xi_0}$$

Eqs. (1)–(3) can be re-written as

$$\eta^4 \frac{d^2\psi}{d\eta^2} = \exp(-\psi) \tag{16}$$

$$\psi(\eta_0) = 0 \tag{17}$$

$$\left.\frac{d\psi}{d\eta}\right|_{\eta=\eta_0} = 0 \tag{18}$$

As a first approximation for large ξ_0 or small η_0, we can approximate equation (16) as

$$\eta_0^4 \frac{d^2\psi}{d\eta^2} = \exp(-\psi) \tag{19}$$

The solution of (19) subject to (17) and (18) is

$$\psi = 2\ln\left(\cosh\left(\frac{\eta - \eta_0}{\eta_0^2\sqrt{2}}\right)\right) = 2\ln\left(\cosh\left(\frac{\xi_0}{\xi}\,\frac{\xi - \xi_0}{\sqrt{2}}\right)\right) = 2\ln\left(\cosh\left(\lambda\frac{y}{\sqrt{2}}\right)\right) \tag{20}$$

This solution approaches that of a slab as λ approaches 1

Another way for writing down the governing equation is as follows;

$$\frac{d^2\psi}{d\xi^2} + \frac{2}{\xi}\frac{d\psi}{d\xi} = \frac{1}{\xi}\frac{d^2\xi\psi}{d\xi^2} = \exp(-\psi) \tag{21}$$

Let

$$\omega = \xi\psi$$

Eq. (21) becomes

$$\frac{d^2\omega}{d\xi^2} = \xi \exp\left(-\frac{\omega}{\xi}\right) \tag{22}$$

As a first approximation for the solution of (21) for ξ close to ξ_0, we write

$$\frac{d^2\omega}{d\xi^2} = \xi_0 \exp\left(-\frac{\omega}{\xi_0}\right) \tag{23}$$

Whose solution with the initial conditions (2,3) is

$$\psi = \frac{\omega}{\xi} = \frac{2\xi_0}{\xi}\ln\left(\cosh\left(\frac{\xi-\xi_0}{\sqrt{2}}\right)\right) = 2\lambda \ln\left(\cosh\left(\frac{y}{\sqrt{2}}\right)\right) \tag{24}$$

3. Development of the approximate solution

We present in this section approximate solution which mainly satisfies the perturbation solution (11) to a certain degree. In addition to the perturbation solution (11), the approximate solution will contain all solutions presented in the previous section.

First we have the approximate solution which is third order accurate and is given by Eq. (13)

$$\psi = \ln\left(1+\frac{y^2}{2}\left(1-\frac{2}{3.\sqrt[4]{1+y^2/15}}\cos\left(\frac{\sqrt{7}}{4}\ln\left(1+y^2/as\right)\right)\right)\right) + \lambda\ln\left(1+y^2\left(1-\frac{2}{3.\sqrt[4]{1+2y^2/15}}\cos\left(\frac{\sqrt{7}}{4}\ln\left(1+y^2/az\right)\right)\right)\right) \tag{13}$$

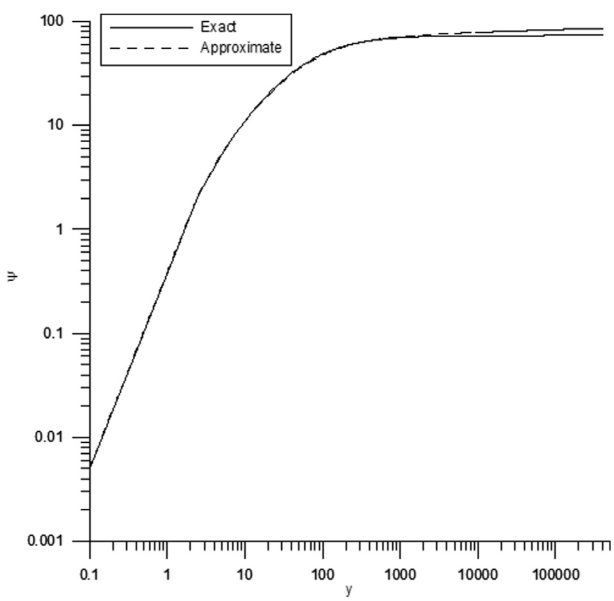

Fig. 7. Ψ against y for ξ_0 = 50

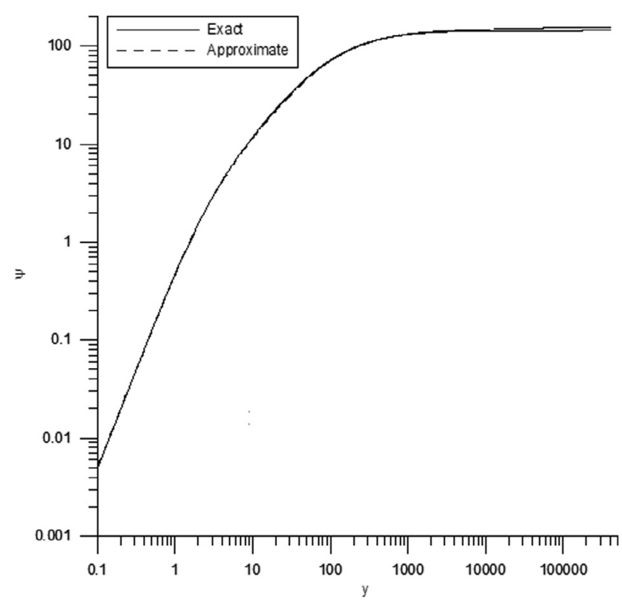

Fig. 9. Ψ against y for ξ_0 = 100

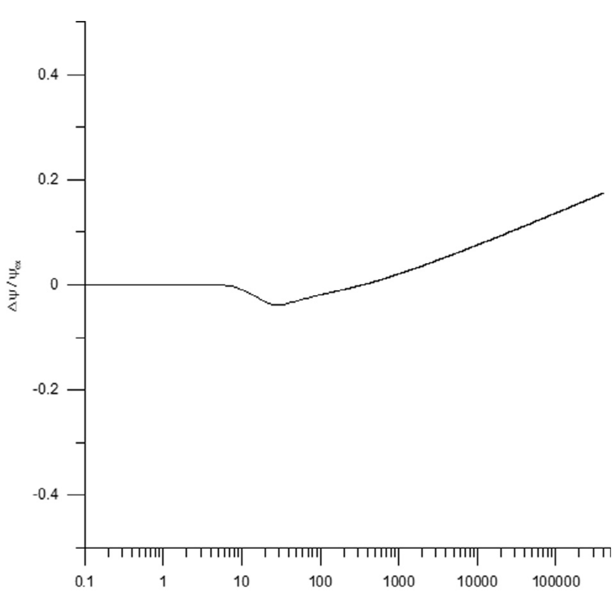

Fig. 8. Relative error in Ψ against y for the approximation for ξ_0 = 50

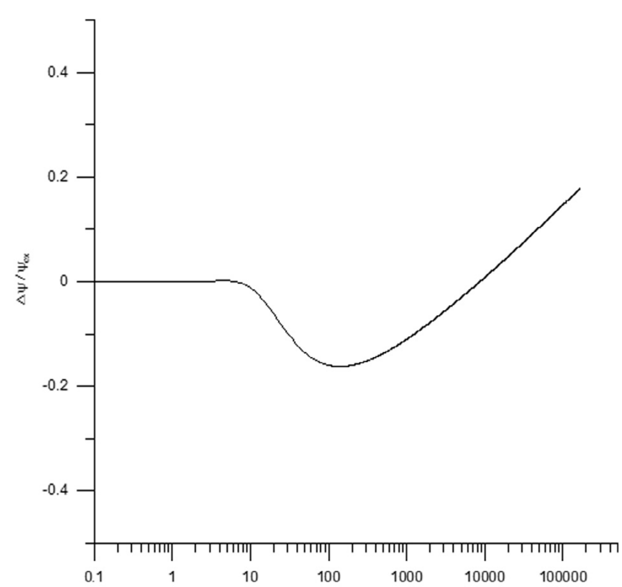

Fig. 10. Relative error in Ψ against y for the approximation for ξ_0 = 100

$$as = \sqrt{(63 * 105/13)}$$

$$az = \sqrt{(63 * 105/4)}$$

Numerical testing with Eq. (13) indicated that it is only accurate for small y.

Attempts have been made to combine the third order approximation with the asymptotic solutions presented in Section 2. These attempts resulted in the following approximation;

$$\begin{aligned}
q &= 1 + \frac{1 + 0.52\lambda y}{1 + \lambda y} \\
\psi_1 &= (1 - \lambda^q)\ln\left(1 + \frac{y^2}{2}\left(1 - \frac{2}{3.\sqrt[4]{1 + y^2/15}}\cos\left(\frac{\sqrt{7}}{4}\ln(1 + y^2/as)\right)\right)\right) \\
&\quad + (\lambda - \lambda^q)\ln\left(1 + y^2\left(1 - \frac{2}{3.\sqrt[4]{1 + 2y^2/15}}\cos\left(\frac{\sqrt{7}}{4}\ln(1 + y^2/az)\right)\right)\right) \\
\psi_2 &= 2\lambda^q \ln\left(\cosh\left(\frac{y}{\sqrt{2}}\right)\right) \\
\psi_3 &= (1 - \lambda^q)\ln(1 + \frac{\lambda^2 y^2}{2}\left(1 - \frac{2}{3.\sqrt[4]{1 + \lambda^2 y^2/15}}\cos\left(\frac{\sqrt{7}}{4}\ln\left(1 + \lambda^2 y^2/as\right)\right)\right) \\
&\quad + (\lambda - \lambda^q)\ln\left(1 + \lambda^2 y^2\left(1 - \frac{2}{3.\sqrt[4]{1 + 2\lambda^2 y^2/15}}\cos\left(\frac{\sqrt{7}}{4}\ln\left(1 + \lambda^2 y^2/az\right)\right)\right)\right) \\
\psi_4 &= \psi_2 + \psi_3 \\
\psi_5 &= 0.00001\lambda^4 y^4\left(\frac{2\ln\left(\cosh\left(\frac{\lambda y}{\sqrt{2}}\right)\right) - \psi_4}{1 + 0.00001\lambda^4 y^4}\right) \\
\psi &= \psi_1 + \psi_2 + \psi_5
\end{aligned} \tag{25}$$

This solution has the property that as the annulus radius increases such that ξ_0 tends to ξ, the solution converges to that of a slab given by Eq. (4).

4. Numerical results and discussion

Exact numerical results were obtained using DASSL FORTRAN code (Petzold, 1982) to solve the system equations (1)–(3). It uses backward differentiation formula methods to solve a system of differential algebraic equations.

Figs. 1, 3, 5, 7 and 9 are plots of ψ against ξ for the numerically exact solution and equation (25) for the cases of ξ_0 = 1,10,20,50,and 100 respectively and Figs. 2, 4, 6, 8, Fig. 10 are the corresponding figures for the relative error in ψ against y for Eq. (25). We notice that the maximum relative error in ψ occurs for ξ_0 = 10 and reaches about −20% at intermediate values of y of about 200 and is much less than this value for other values of y and ξ_0.

The maximum relative error decreases as ξ_0 increase beyond ξ_0 = 10. Improvement in model parameters is needed in the intermediate range of y and ξ_0.

5. Conclusions

Approximate analytical solution for the isothermal Lane-Emden equation in a spherical annulus was derived. This solution is of good accuracy for small annulus radius ξ_0. Eq. (25) was of good accuracy for all values of ξ and ξ_0 except for small intermediate region. It can be improved by modifying the equation parameters. Extension of this work to the corresponding boundary value problem of Frank Kamenentskii is currently under investigation. This corresponds to thermal ignition and zero order catalytic reaction in a hollow sphere.

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